q-Discrete Painlevé equations for recurrence coefficients of modified q-Freud orthogonal polynomials

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August 7, 2008

Abstract

We present an asymmetric q-Painlevé equation. We will derive this using q-orthogonal polynomials with respect to generalized Freud weights: their recurrence coefficients will obey this q-Painlevé equation (up to a simple transformation). We will show a stable method of computing a special solution which gives the recurrence coefficients. We establish a connection between the newfound equation and α-q-P_{V}.

1 Introduction

The aim of this work is to identify new q-discrete Painlevé equations. What exactly are the discrete Painlevé equations (d-P)? One could state that a d-P is a second-order, nonautonomous integrable mapping having one of the celebrated Painlevé equations for a continuous limit. This description does not quite capture the whole story, so let's look at some important notions about discrete Painlevé equations.

Although some discrete Painlevé equations were found *avant la lettre*, the first steps were made by Brézin and Kazakov [1] who found an equation (now known as d-P_{I}) and computed its continuous limit, the continuous Painlevé I. The definition of *singularity confinement* [5] was maybe the most important step in the evolution of the field: it is the discrete analogue of the Painlevé property and describes the behaviour of singularities throughout the evolution of a discrete equation:

Definition 1.1 (Singularity Confinement Property). *Consider a difference equation with independent variable n and dependent variable x_n. If x_n is such that it gives rise to a singularity for x_{n+1}, then there exists a $p \in \mathbb{N}$ such that the singularity is confined to $x_{n+1}, \ldots, x_{n+p}$ and x_{n+p+1} depends only on $x_{n-1}, x_{n-2}, \ldots$*

Using only this criterion of singularity confinement, discrete analogues for (continuous) $\mathrm{P}_{\mathrm{III}}$, P_{IV} and P_{V} were identified [5], [9]. It's striking that in the analogue of $\mathrm{P}_{\mathrm{III}}$ we see the independent variable n entering the equation only in an exponential way:

$$x_{n+1}x_{n-1} = \frac{(x_n+\alpha)(x_n+\beta)}{(\gamma q^n+1)(\delta q^n+1)}$$

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with $\alpha, \beta, \gamma, \delta$ constants. Therefore, it is considered to be a so-called *q-discrete* equation as the nature of the equation is multiplicative rather than additive like, e.g., d-$\mathrm{P_I}$

$$x_{n-1} + x_n + x_{n+1} = \frac{z_n + \gamma(-1)^n}{x_n} + \delta$$

with $z_n = \alpha n + \beta$ and $\alpha, \beta, \gamma, \delta$ constants. This equation shows an odd-even dependence through the factor $(-1)^n$ when $\gamma \neq 0$. In this case we can introduce new variables $u_n = x_{2n}$, $v_n = x_{2n+1}$ that lead to the system

$$\begin{cases} v_{n-1} + u_n + v_n &= \frac{2\alpha n+\beta+\gamma}{u_n} + \delta &= \frac{z_n+\gamma}{u_n} + \delta \\ u_n + v_n + u_{n+1} &= \frac{2\alpha(n+1)+\beta-\gamma}{v_n} + \delta &= \frac{z_{n+1}-\gamma}{v_n} + \delta \end{cases}$$

with $z_n = 2n\alpha + \beta$. This system is known as $\alpha-$d-$\mathrm{P_I}$, an *asymmetric* discrete Painlevé equation. A list of a few important discrete Painlevé equations was compiled by Peter Clarkson, and can be found in [11]. See also [4] for an overview of discrete Painlevé equations.

The link between orthonormal polynomials and discrete Painlevé equations is well established ([2], [7], [11]). Given a positive measure w on a set $A \subset \mathbb{R}$ and assuming all the moments for w exist, i.e. $|\int_A x^k w(x)dx| < \infty$, we denote with $\{p_n\}$ the set of orthonormal polynomials with respect to w:

$$\int_A p_n(x)p_m(x)w(x)dx = \delta_{mn} \qquad m, n \geq 0.$$

These orthonormal polynomials are unique if we choose the leading coefficient to be positive. Orthonormal polynomials satisfy a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x) \qquad n \geq 0$$

with $p_{-1} = 0$. Here we can find the recurrence coefficients a_n and b_n as the coefficients showing up in the Fourier expansion of $xp_n(x)$:

$$a_n = \int_A xp_n(x)p_{n-1}(x)w(x)dx, \tag{1.1}$$

$$b_n = \int_A xp_n^2(x)w(x)dx. \tag{1.2}$$

If we write $p_n(x) = \gamma_n x^n + \ldots$, we can find $a_n = \frac{\gamma_{n-1}}{\gamma_n}$ by comparing the leading coefficients of both sides of the recurrence relation. Since we chose to take positive leading coefficients, a_n will be positive as well.

During the last few decades, many generalizations of well-known weight functions have been shown to give rise to discrete Painlevé equations for the recurrence coefficients a_n and b_n.

As far as we can see, the very first appearance of a d-P was in the context of orthogonal polynomials, when Shohat [10] found a nonlinear recurrence relation for the recurrence coefficients, which is now known as d-$\mathrm{P_I}$. Fokas, Its and Kitaev found the connection of d-$\mathrm{P_I}$ to Freud weights in [2]. Nijhoff [8] found a q-discrete Painlevé equation in the context of orthogonal polynomials: he considered a q-generalization of the Hermite polynomials on the exponential lattice and found non-linear recurrences of order >2. A second order recurrence relation was found by the third author [11] which he called q-$\mathrm{P_I}$. We will study a slight extension of this Freud weight to recover asymmetric q-Painlevé equations.

In particular we consider in Section 3 the weight

$$w(x) = \frac{(q^4x^4;q^4)_\infty |x|^\alpha}{(1-q^4)^{\alpha/4}} \qquad \alpha > -1$$

on the q-exponential lattice $L = \{\pm q^n, n \in \mathbb{N}\}$, with $q \in (0,1)$. We will show that $b_n = 0$ and $y_n = a_n^2 q^{1-n}$ satisfies

$$q^{n-\alpha}(y_n y_{n+1} + q^\alpha)(y_n y_{n-1} + q^\alpha) = \begin{cases} q^\alpha - q^{-\alpha} y_n^2 & n \text{ even} \\ 1 - y_n^2 & n \text{ odd} \end{cases}$$

which we will call q-P_I. The case $\alpha = 0$ was already obtained in [11]. In Section 4 we consider the more general weight

$$w(x) = \frac{|x|^\alpha (q^2x^2;q^2)_\infty (cq^2x^2;q^2)_\infty}{(1-q^4)^{\alpha/4}}, \qquad \alpha > -1, c \le 0.$$

We show that $y_n = a_n^2 q^{1-n}$ now satisfies

$$q^{n-\alpha}(-cy_n y_{n+1} + q^\alpha)(-cy_n y_{n-1} + q^\alpha) = \begin{cases} (q^\alpha - y_n)(q^\alpha - cy_n)q^{-\alpha} & n \text{ even} \\ (1-y_n)(1-cy_n) & n \text{ odd.} \end{cases}$$

In Section 5 we relate these equations to α-q-$\mathrm{P_V}$. In Section 6 we show a stable method of computing the recurrence coefficients.

2 Preliminaries

We consider orthogonal polynomials on the exponential lattice

$$L = \{\pm q^n | n \in \mathbb{N}\}, \qquad 0 < q < 1.$$

The orthonormality condition with respect to a weight w on L is,

$$\int_{-1}^{1} p_n(x) p_m(x) w(x) d_q x = \delta_{mn}. \tag{2.1}$$

The q-integral is defined as the sum

$$\int_{-1}^{1} f(x) d_q x = (1-q)\sum_{k=0}^{\infty} f(q^k) q^k + (1-q)\sum_{k=0}^{\infty} f(-q^k) q^k.$$

The q-difference operator D_q will play an important role in our results:

$$D_q f(x) = \begin{cases} \dfrac{f(qx) - f(x)}{x(q-1)} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0. \end{cases}$$

We will consider even weights $w(x) = w(-x)$. This implies that the orthonormal polynomials p_n associated with the weight w will satisfy the symmetry property

$$p_n(-x) = (-1)^n p_n(x). \tag{2.2}$$

The orthonormal polynomials of even degree will be even, those of odd degree will be odd. The recurrence relation will then take the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x), \tag{2.3}$$

i.e., all coefficients b_n are equal to zero.

We will use the following technique (see, e.g., [11]) to identify q-discrete Painlevé equations:

- Given an even weight w, let p_n denote the orthonormal polynomials with respect to w.
- Find the Fourier expansion of the polynomial $D_q p_n$.
- Compare the coefficients of this polynomial and its Fourier expansion. This will give rise to a set of equations.
- After a change of variable, we find a q-discrete Painlevé equation. We check the property of singularity confinement, and try to find out what happens when $q \to 1$.

3 Modified q-Freud polynomials

We consider the weight

$$w(x) = \frac{(q^4x^4;q^4)_\infty |x|^\alpha}{(1-q^4)^{\alpha/4}} \tag{3.1}$$

on the q-exponential lattice L, with $\alpha > -1$. The q-Pochhammer symbol is defined as

$$(a;q)_n = \prod_{j=0}^{n-1} \left(1 - aq^j\right), \qquad (a;q)_\infty = \prod_{j=0}^{\infty} \left(1 - aq^j\right).$$

Observe that, in terms of the q-exponential function $E_q(z) = (-z;q)_\infty$, we have

$$w\left(\sqrt[4]{1-q^4}x\right) = |x|^\alpha E_{q^4}\left(-(1-q^4)q^4x^4\right)$$

and hence $w(\sqrt[4]{1-q^4}x) \to |x|^\alpha e^{-x^4}$ when $q \to 1$ (see, e.g., [6]). This limit relation is the only reason for the presence of the (constant) denominator in w. So this weight can be called a q-analogue of the modified Freud weight $|x|^\alpha e^{-x^4}$. It is easy to check that this weight satisfies the Pearson equation

$$w\left(\frac{x}{q}\right) = \frac{(1-x^4)}{q^\alpha} w(x). \tag{3.2}$$

Let p_n be the orthonormal polynomials associated to w as in (2.1). Because of (2.2) we can put $p_n(x) = \gamma_n x^n + \delta_n x^{n-2} + ...$, and we have the following

Lemma 3.1. *For orthogonal polynomials with an even weight one has*

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} \qquad \textit{and} \qquad -\sum_{j=1}^{n-1} a_j^2 = \frac{\delta_n}{\gamma_n}.$$

Proof. These two equations can be obtained by comparing, respectively, the x^{n+1} and x^{n-1} terms in the recurrence relation (2.3). □

Lemma 3.2. *For even n, the polynomials p_n with weight (3.1) satisfy the following structure relation:*

$$D_q p_n(x) = \frac{B_n}{1-q} p_{n-1}(x) + \frac{A_n}{1-q} p_{n-3}(x) \tag{3.3}$$

with

$$A_n = \frac{a_n a_{n-1} a_{n-2}}{q^{\alpha+n-3}} \tag{3.4}$$

$$B_n = \frac{a_n}{q^{\alpha+n-1}} \left(\sum_{j=1}^{n+1} a_j^2 - q^2 \sum_{j=1}^{n-2} a_j^2 \right). \tag{3.5}$$

For odd n, the polynomials p_n satisfy

$$D_q p_n(x) = \frac{B_n}{1-q} p_{n-1}(x) + \frac{A_n}{1-q} p_{n-3}(x) + \text{lower order terms} \tag{3.6}$$

with

$$A_n = \frac{a_n a_{n-1} a_{n-2}}{q^{\alpha+n-3}} - \left(1 - q^{-\alpha}\right) \frac{a_{n-1}}{a_n a_{n-2}} \tag{3.7}$$

$$B_n = \frac{a_n}{q^{\alpha+n-1}} \left(\sum_{j=1}^{n+1} a_j^2 - q^2 \sum_{j=1}^{n-2} a_j^2 \right) + \frac{1-q^{-\alpha}}{a_n}. \tag{3.8}$$

Remark 3.3. *Notice the difference between the closed expression of the structure relation for even n, and the presence of all even lower degree polynomials for odd n. Despite this difference, the resulting Painlevé equations for even and odd n will have a very similar structure.*

Proof. If we expand $D_q p_n$ into a Fourier series, we obtain

$$D_q p_n(x) = \sum_{j=0}^{n-1} a_{j,n} p_j(x)$$

with

$$a_{j,n} = \int_{-1}^{1} D_q p_n(x) p_j(x) w(x) d_q(x).$$

The symmetry relation (2.2) shows that $a_{j,n} = 0$ if $n-j$ is even. For $n-j$ odd we get

$$\begin{aligned} a_{j,n} &= 2(1-q) \sum_{k=0}^{\infty} D_q p_n(q^k) p_j(q^k) w(q^k) q^k \\ &= -2 \sum_{k=0}^{\infty} \left(p_n(q^{k+1}) - p_n(q^k)\right) p_j(q^k) w(q^k) \\ &= -2 \sum_{k=0}^{\infty} p_n(q^{k+1}) p_j(q^k) w(q^k) + 2 \sum_{k=0}^{\infty} p_n(q^k) p_j(q^k) w(q^k). \end{aligned}$$

Since $n-j$ is odd, (2.2) implies that both sums are finite. Now we perform a shift in the summation index of the first sum, so that it contains the expression $w(q^{k-1})$, and we apply the Pearson equation (3.2) on this. We can recognize all sums as q-integrals, and we obtain

$$\begin{aligned} a_{j,n} &= -\frac{q^{-\alpha}}{1-q} \int_{-1}^{1} p_n(x) p_j(x/q) \frac{w(x)}{x} d_q x + \frac{q^{-\alpha}}{1-q} \int_{-1}^{1} p_n(x) p_j(x/q) x^3 w(x) d_q x \\ &\quad + \frac{1}{1-q} \int_{-1}^{1} p_n(x) p_j(x) \frac{w(x)}{x} d_q x. \end{aligned} \tag{3.9}$$

From now on we have to make a distinction based on the parity of n.

Case 1: n is even. Since $n-j$ is odd, we know from (2.2) that $p_j(x)$ and $p_j(x/q)$ are odd polynomials, hence $\frac{p_j(x)}{x}$ and $\frac{p_j(x/q)}{x}$ are polynomials, and the orthogonality relations (2.1) imply that the first and the third integral in (3.9) vanish. To obtain the value of the second integral, we write $p_j(x/q)x^3$ as a linear combination of the orthonormal polynomials $p_k(x)$: an easy calculation shows that

$$p_j(x/q)x^3 = \frac{\gamma_j}{\gamma_{j+3}q^j}p_{j+3}(x) + \left(\frac{\delta_j}{\gamma_{j+1}q^{j-2}} - \frac{\gamma_j\delta_{j+3}}{\gamma_{j+3}\gamma_{j+1}q^j}\right)p_{j+1}(x) + \text{lower order terms}$$

Using Lemma 3.1 and the orthonormality relations (2.1) we obtain (3.3)-(3.5).

Case 2: n is odd. The second integral in (3.9) can be computed in exactly the same way as in the previous case. However, this time the contributions of the first and the third integral do not vanish. For the third integral, we need to write $p_n(x)/x$ as a linear combination of the orthonormal polynomials $p_k(x)$. This can be done because n is odd, and hence p_n is an odd polynomial. This yields

$$\frac{p_n(x)}{x} = \frac{\gamma_n}{\gamma_{n-1}}p_{n-1}(x) + \left(\frac{\delta_n}{\gamma_{n-3}} - \frac{\gamma_n\delta_{n-1}}{\gamma_{n-1}\gamma_{n-3}}\right)p_{n-3}(x) + \text{lower order terms}$$

As for the first integral in (3.9), we can use the orthogonality relations (2.1) to write

$$\begin{aligned}\int_{-1}^{1} p_n(x)p_j(x/q)\frac{w(x)}{x}d_qx &= \int_{-1}^{1} p_n(x)\frac{p_j(x/q)-p_j(x)+p_j(x)}{x}w(x)d_qx \\ &= \int_{-1}^{1} p_n(x)p_j(x)\frac{w(x)}{x}d_qx,\end{aligned}$$

which is the same as the third integral in (3.9). Combining these results with Lemma 3.1 we obtain (3.6)-(3.8). □

Now we can use these structure relations (3.3)-(3.8) to obtain relations between the recurrence coefficients a_n. Comparing coefficients of x^{n-1} and x^{n-3} in these structure relations and using Lemma 3.1 gives

$$a_n^2a_{n-1}^2a_{n-2}^2 = q^{n+\alpha-3}\left(q^{n-2}(1-q^2)\sum_{j=1}^{n-2}a_j^2 - (1-q^{n-2})a_{n-1}^2\right) \tag{3.10}$$

and

$$a_n^2\left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 + (1-q^2)\sum_{j=1}^{n-2}a_j^2\right) = (1-q^n)q^{\alpha+n-1} \tag{3.11}$$

for even n, and

$$q^{-\alpha-n+3}a_n^2a_{n-1}^2a_{n-2}^2 = q^{n-2}(1-q^2)\sum_{j=1}^{n-2}a_j^2 - \left(q^{-\alpha}-q^{n-2}\right)a_{n-1}^2 \tag{3.12}$$

and

$$a_n^2\left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 + (1-q^2)\sum_{j=1}^{n-2}a_j^2\right) = \left(q^{-\alpha}-q^n\right)q^{\alpha+n-1} \tag{3.13}$$

for odd n. The aim is to obtain a Painlevé-type recurrence relation between a_{n+1}, a_n and a_{n-1}. Hence to get rid of the a_{n-2} in (3.10) and (3.12), we replace n by $n+1$. Keep in mind

that this changes the parity of n. The sum $\sum_{j=1}^{n-1} a_j^2$ that arises in this way, should then be seen as $\sum_{j=1}^{n-2} a_j^2 + a_{n-1}^2$, where we can use (3.13) and (3.11) respectively to write these sums as a function of a_{n+1}, a_n and a_{n-1} only. These manipulations yield the following recurrence relations:

$$\begin{cases} a_n^2\left(a_{n+1}^2 + q^{-\alpha-n+1}a_n^2 + q^2 a_{n-1}^2 + q^{-2n-\alpha+3}a_{n+1}^2 a_n^2 a_{n-1}^2\right) = (1-q^n)\, q^{\alpha+n-1}, & n \text{ even} \\ a_n^2\left(a_{n+1}^2 + q^{-n+1}a_n^2 + q^2 a_{n-1}^2 + q^{-2n-\alpha+3}a_{n+1}^2 a_n^2 a_{n-1}^2\right) = (q^{-\alpha}-q^n)\, q^{\alpha+n-1}, & n \text{ odd.} \end{cases} \tag{3.14}$$

This is a q-deformation of the discrete Painlevé I equation d-P$_\mathrm{I}$, which in its most general form is given by

$$x_{n+1} + x_n + x_{n-1} = \frac{an+b+c(-1)^n}{x_n} + d. \tag{3.15}$$

Indeed, if we take $x_n = a_n^2/\sqrt{1-q^4}$ and we let q tend to 1, we get

$$x_{n+1} + x_n + x_{n-1} = \frac{2n+\alpha-\alpha(-1)^n}{8x_n}.$$

Putting $y_n = a_n^2 q^{1-n}$, we can factorize the equations: we obtain

$$q^{n-\alpha}\left(y_n y_{n+1} + q^\alpha\right)\left(y_n y_{n-1} + q^\alpha\right) = \begin{cases} q^\alpha - q^{-\alpha} y_n^2, & \text{for even } n \\ 1 - y_n^2 & \text{for odd } n. \end{cases} \tag{3.16}$$

If we write $u_n = q^{-\alpha} y_{2n}$ and $v_n = -y_{2n+1}$, substituting n by $2m$ resp. $2m+1$ in (3.16), we get

$$\begin{cases} q^{2m}\left(1-u_m v_m\right)\left(1-u_m v_{m-1}\right) &= 1-u_m^2 \\ q^{2m+1+\alpha}\left(1-u_m v_m\right)\left(1-u_{m+1} v_m\right) &= 1-v_m^2. \end{cases} \tag{3.17}$$

This set of equations could therefore be called a set of asymmetric q-discrete Painlevé I equations (α-q-P$_\mathrm{I}$). However, in the next section we will obtain a more general form of it. Concerning the asymptotic behaviour of a_n as n tends to infinity (with fixed q), it follows easily from (3.14) that a_n tends to 0, and that

$$\lim_{n\to\infty} \frac{a_{2n}^2}{q^{2n-1}} = \lim_{n\to\infty} y_{2n} = q^\alpha \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_{2n+1}^2}{q^{2n}} = \lim_{n\to\infty} y_{2n+1} = 1, \tag{3.18}$$

or, equivalently,

$$\lim_{n\to\infty} u_n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} v_n = -1.$$

The set of equations (3.16) has the singularity confinement property: if $y_n = 0$, then a singularity occurs for y_{n+1} but this singularity does not propagate infinitely: e.g. for n even, putting $y_n = \epsilon$ we obtain

$$\begin{aligned} y_n &= \varepsilon \\ y_{n+1} &= -q^\alpha\left(1-q^{-n}\right)\frac{1}{\varepsilon} - y_{n-1}q^{-n} + \mathcal{O}(\varepsilon) \\ y_{n+2} &= q^{\alpha-1}\left(1-q^{-n}\right)\frac{1}{\varepsilon} + \frac{y_{n-1}}{q} + \mathcal{O}(\varepsilon) \\ y_{n+3} &= \frac{q^{n+1}-q^{-\alpha-2}}{1-q^n}\varepsilon + \mathcal{O}(\varepsilon^2) \\ y_{n+4} &= q^{\alpha+2}\frac{1-q^n}{1-q^{n+\alpha+3}} y_{n-1} + \mathcal{O}(\varepsilon) \end{aligned}$$

so the singularity is confined to $y_{n+1}, y_{n+2}, y_{n+3}$; furthermore y_{n+4} depends on the value y_{n-1} before the singularity. The same holds for odd n:

$$\begin{aligned}
y_n &= \varepsilon \\
y_{n+1} &= \left(q^{-n} - q^{\alpha}\right) \frac{1}{\varepsilon} - y_{n-1} q^{-n-\alpha} + \mathcal{O}(\varepsilon) \\
y_{n+2} &= \left(q^{\alpha-1} - q^{-n-1}\right) \frac{1}{\varepsilon} + \frac{y_{n-1}}{q} + \mathcal{O}(\varepsilon) \\
y_{n+3} &= q^{\alpha} \frac{q^{n} - q^{-3}}{1 - q^{n+\alpha}} \varepsilon + \mathcal{O}(\varepsilon^2) \\
y_{n+4} &= q^{2-\alpha} \frac{1 - q^{n+\alpha}}{1 - q^{n+3}} y_{n-1} + \mathcal{O}(\varepsilon).
\end{aligned}$$

There are two (one for each parity) more critical cases that might give rise to singularities, namely when $y_n y_{n-1} + q^{\alpha} = 0$. Running the singularity analysis gives, for even n,

$$\begin{aligned}
y_n &= -\frac{q^{\alpha}}{y_{n-1}} + \varepsilon \\
y_{n+1} &= \frac{q^{-n+\alpha}(y_{n-1}^2 - 1)}{y_{n-1}^2} \frac{1}{\varepsilon} + \frac{q^{-n} + q^{-n} y_{n-1}^2 - y_{n-1}^2}{y_{n-1}} + \mathcal{O}(\varepsilon)
\end{aligned}$$

The coefficient of $1/\varepsilon$ however, is $\mathcal{O}(\varepsilon)$ itself as y_{n-1} satisfies

$$q^{n-\alpha-1} \left(y_n y_{n-1} + q^{\alpha}\right) \left(y_{n-2} y_{n-1} + q^{\alpha}\right) = 1 - y_{n-1}^2$$

or, after substituing y_n,

$$1 - y_{n-1}^2 = \varepsilon y_{n-1} q^{n-1-\alpha} (y_{n-1} y_{n-2} + q^{\alpha}).$$

So there is no singularity to confine.

One could hope to use the recurrence relations (3.16) to compute the recurrence coefficients. Given $y_0 = 0$, it would be nice to show that there is a unique solution to (3.16) which is positive for all $n > 0$. The existence of such a solution is clear since $y_n = a_n^2 / q^{n-1}$ satisfies the given recurrence relations and is obviously positive for $n > 0$. Its uniqueness however can only be expected based on numerical experiments, but we have no proof of it yet. The method used in Section 6 is probably the key to proving this uniqueness. Using (1.1) and the q-binomial theorem, it is clear that

$$y_1 = a_1^2 = \frac{\int_{-1}^{1} x^2 \left(x^4 q^4; q^4\right)_{\infty} |x|^{\alpha} d_q x}{\int_{-1}^{1} \left(x^4 q^4; q^4\right)_{\infty} |x|^{\alpha} d_q x} = \frac{(q^{\alpha+1}; q^4)_{\infty}}{(q^{\alpha+3}; q^4)_{\infty}}.$$

The knowledge of y_0 and y_1 and the recurrence relation (3.16) allow us to compute the y_n recursively. However, this turns out to be an unstable method. We computed y_n to a certain accuracy for some particular choices of q and α. In Figure 1 we plotted the values of $\log |y_n|$, calculated with an accuracy of 200 digits, with parameters $q = 0.9, \alpha = 5$. For this choice of parameters, the computed values seem to satisfy the limit behaviour as stated in (3.18) up to $n \sim 90$, but for larger n the sensibility of the non-linear equations on the initial values destroys this behaviour. This is why Section 6 will be devoted to a more stable way of computing the recurrence coefficients.

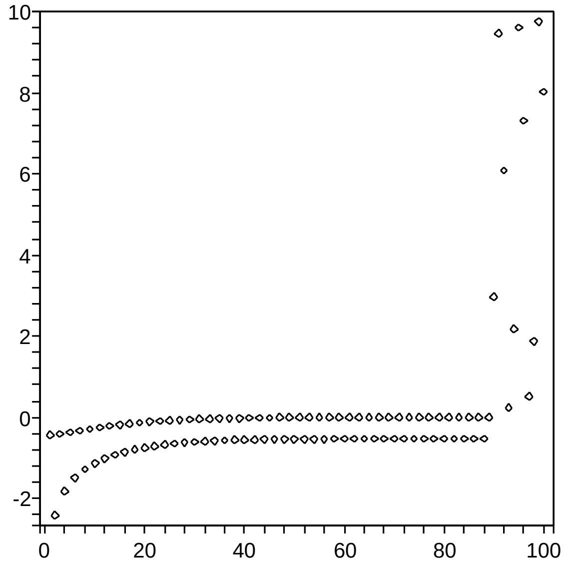

Figure 1: The result of computing $\log |y_n|$ from (3.16) with $\alpha = 5$, $q = 0.9$, with an accuracy of 200 digits. The odd-even dependence is clearly visible.

4 Another modified discrete q-Freud case

In this section we consider a generalization of the previous case: let the weight be given by

$$w(x) = \frac{|x|^\alpha \left(q^2x^2;q^2\right)_\infty \left(cq^2x^2;q^2\right)_\infty}{\left(1-q^4\right)^{\alpha/4}} \tag{4.1}$$

with $c \leq 0$. This weight satisfies the Pearson equation

$$w\left(\frac{x}{q}\right) = \frac{\left(1-x^2\right)\left(1-cx^2\right)}{q^\alpha} w(x).$$

If

$$c = -1 + a\sqrt{1-q^4} \tag{4.2}$$

then

$$w\left(\sqrt[4]{1-q^4}\,x\right) \to |x|^\alpha e^{-x^4-2ax^2} \qquad \text{as } q \to 1,$$

so this weight is a q-analogue of the Freud weight $|x|^\alpha e^{-x^4-2ax^2}$. Obviously the choice $c = -1$ gives us the case considered in the previous section. The choice $c = 0$ leaves us with a q-analogue of the weight $|x|^\alpha e^{-x^2}$, which was already studied in [11] in the case $\alpha = 0$.

Lemma 4.1. *For even n, the orhogonal polynomials p_n for the weight w in (4.1) satisfy the following structure relation:*

$$D_q p_n(x) = \frac{\hat{B}_n}{1-q} p_{n-1}(x) + \frac{\hat{A}_n}{1-q} p_{n-3}(x)$$

with

$$\hat{A}_n = -c\frac{a_n a_{n-1} a_{n-2}}{q^{\alpha+n-3}}$$

$$\hat{B}_n = \frac{a_n}{q^{\alpha+n-1}} \left(c + 1 - c \sum_{j=1}^{n+1} a_j^2 + cq^2 \sum_{j=1}^{n-2} a_j^2 \right).$$

For odd n, the polynomials p_n satisfy

$$D_q p_n(x) = \frac{\hat{B}_n}{1-q} p_{n-1}(x) + \frac{\hat{A}_n}{1-q} p_{n-3}(x) + \text{lower order terms}$$

with

$$\hat{A}_n = -c \frac{a_n a_{n-1} a_{n-2}}{q^{\alpha+n-3}} - \left(1 - q^{-\alpha}\right) \frac{a_{n-1}}{a_n a_{n-2}}$$

$$\hat{B}_n = -c \frac{a_n}{q^{\alpha+n-1}} \left(\sum_{j=1}^{n+1} a_j^2 - q^2 \sum_{j=1}^{n-2} a_j^2 \right) + \frac{1 - q^{-\alpha}}{a_n} + (c+1) \frac{a_n}{q^{\alpha+n-1}}.$$

Proof. The proof follows the same steps as the proof of Lemma 3.2. The only difference is that in the Pearson equation, the quartic polynomial $1 - x^4$ has now got to be replaced by $1 - (c+1)x^2 + cx^4$. $\square$

Performing the same manipulations as in the previous section leads us to a recurrence relation satisfied by the a_n:

$$\begin{cases} a_n^2 \left(c + 1 - c \left(a_{n+1}^2 + q^{-\alpha-n+1} a_n^2 + q^2 a_{n-1}^2 - cq^{-2n-\alpha+3} a_{n+1}^2 a_n^2 a_{n-1}^2\right)\right) = \left(1 - q^n\right) q^{\alpha+n-1}, & n \text{ even} \\ a_n^2 \left(c + 1 - c \left(a_{n+1}^2 + q^{-n+1} a_n^2 + q^2 a_{n-1}^2 - cq^{-2n-\alpha+3} a_{n+1}^2 a_n^2 a_{n-1}^2\right)\right) = \left(q^{-\alpha} - q^n\right) q^{\alpha+n-1}, & n \text{ odd.} \end{cases} \tag{4.3}$$

Putting again $y_n = a_n^2 q^{1-n}$ we obtain a q-discrete Painlevé I equation, of which (3.16) is a special case:

$$q^{n-\alpha} \left(-cy_n y_{n+1} + q^\alpha\right) \left(-cy_n y_{n-1} + q^\alpha\right) = \begin{cases} \left(q^\alpha - y_n\right) \left(q^\alpha - cy_n\right) q^{-\alpha}, & \text{for even } n \\ \left(1 - y_n\right) \left(1 - cy_n\right), & \text{for odd } n. \end{cases} \tag{4.4}$$

We can rewrite these equations into a more familiar form for asymmetric discrete Painlevé equations: denoting $u_n = y_{2n} q^{-\alpha}$ and $v_n = cy_{2n+1}$ we get

$$\begin{cases} q^{2n} \left(1 - u_n v_n\right) \left(1 - u_n v_{n-1}\right) &= \left(1 - u_n\right) \left(1 - cu_n\right) \\ q^{2n+1+\alpha} \left(1 - u_n v_n\right) \left(1 - u_{n+1} v_n\right) &= \left(1 - v_n\right) \left(1 - v_n/c\right). \end{cases} \tag{4.5}$$

Besides q, this set of equations contains the parameters α and c, and it is the most general form of the asymmetric q-discrete Painlevé I equations we obtain. Hence we call this set α-q-$\mathrm{P_I}$. It is interesting to notice that these modifications w.r.t. the previous section 'survive' in the limit case $q \to 1$: if we put $x_n = \frac{a_n^2}{\sqrt{1-q^4}}$ and we let q tend to 1, we obtain

$$x_{n+1} + x_n + x_{n-1} = \frac{2n + \alpha - \alpha(-1)^n}{8x_n} - a,$$

which is still an instance of the discrete Painlevé I equation (3.15). Here a and c are related as in (4.2). The introduction of the additional parameter c has no influence on the limit behaviour of the recurrence coefficients a_n: (3.18) still holds. To show this we distinguish the following cases. We only mention the proof for even n, the odd n case being analogous.

- $c = -1$ is the case from the previous section.

- $0 > c > -1$: for even n, (4.3) gives

$$\frac{a_n^4}{q^{2n-2}} \leq \frac{q^{2\alpha}(1-q^n)}{-c}$$

so $a_n \to 0$ and $y_n = \frac{a_n^2}{q^{n-1}}$ is bounded. Denoting $A = \limsup y_n$, (4.3) gives the quadratic equation $A\left(c+1-\frac{cA}{q^\alpha}\right) = q^\alpha$. This equation has a negative solution $A = q^\alpha/c$ which we can reject, and a positive solution $A = q^\alpha$. The same argument can be used on $B = \liminf y_n$, and one obtains (3.18).

- $c < -1$, n even: from (4.3) we get

$$\frac{a_n^2}{q^{\alpha+n-1}}\left(c+1-c\frac{a_n^2}{q^{\alpha+n-1}}\right) \leq 1-q^n < 1.$$

Looking at this as a quadratic inequality for $y_n = \frac{a_n^2}{q^{n-1}}$ we obtain

$$\frac{y_n}{q^\alpha}\left(c+1-\frac{cy_n}{q^\alpha}\right) < 1$$

and hence $y_n \in (1/c, 1)$. A similar argument with lim sup and lim inf yields (3.18).

- $c = 0$: then the result follows immediately from (4.3).

Once more, the singularity confinement property is fulfilled: taking $y_n = \epsilon$ for an even n, one finds that y_{n+1} and y_{n+2} are $\mathcal{O}(1/\varepsilon)$, y_{n+3} is $\mathcal{O}(\varepsilon)$ and

$$y_{n+4} = \frac{q^{2+\alpha}}{c\,(1-q^{n+\alpha+3})}\left[cy_{n-1}(1-q^n)-(1+c)(1-q^{-2})\right]+\mathcal{O}(\epsilon).$$

Now it is not obvious that the constant term is nonzero: if $y_{n-1} = \frac{(1+c)(1-q^{-2})}{c(1-q^n)}$ then this is not yet a proof that the singularity confinement property holds. For odd n a similar argument yields

$$y_{n+4} = \frac{q^{2-\alpha}}{c\,(1-q^{n+3})}\left[cy_{n-1}(1-q^{n+\alpha})-(1+c)(1-q^{-2})q^\alpha\right]+\mathcal{O}(\epsilon)$$

and a problem could arise if $y_{n-1} = \frac{q^\alpha(1+c)(1-q^{-2})}{c(1-q^{n+\alpha})}$. However (consider the case n is even, the odd n case being completely analogous) the zero at y_{n+4} gives rise to singularities in $y_{n+5}, y_{n+6}, y_{n+7}$, but the singularity vanishes in y_{n+8}. Moreover, for the same reasons as above, this y_{n+8} can only be zero if $y_{n+3} = \frac{1+c}{c}\frac{1-q^{-2}}{1-q^{n+4}}$ (which is nonzero since $|q| < 1$ and $c \neq -1$, the case $c = -1$ being considered in the previous section), while the computation gave $y_{n+3} = \mathcal{O}(\epsilon)$. So this assures us that even in the worst case, the singularity is confined to $y_{n+1}, \ldots, y_{n+7}$. Again, as in Section 3, there are no singularities arising from $-cy_ny_{n-1}+q^\alpha = 0$ due to fine cancellations. Concerning the use of the recurrence relations to compute the recurrence coefficients a_n of the orthogonal polynomials, we now start with $y_0 = 0$ and

$$y_1 = a_1^2 = \frac{\sum_{k=0}^\infty \frac{q^{k(\alpha+3)}}{(q^2;q^2)_k(cq^2,q^2)_k}}{\sum_{k=0}^\infty \frac{q^{k(\alpha+1)}}{(q^2;q^2)_k(cq^2,q^2)_k}}.$$

The same remark as in the previous section holds: this method is unstable, as is shown in Figure 2. We refer to Section 6 for a stable computation method.

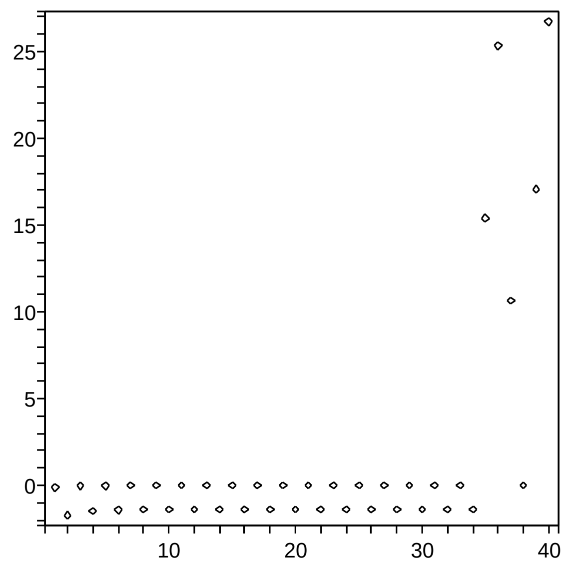

Figure 2: The result of computing $\log |y_n|$ from (4.4) with $\alpha = 2$, $q = 0.5$, $c = -1/3$, with an accuracy of 200 digits.

Remark 4.2. *We restricted ourselves to the choice $c < 0$, with the case $c = 0$ being a limiting case where the Painlevé equations give explicit expressions for e.g. y_n. However, numerical experiments strongly suggest that also for $0 < c < 1$ (in which case the weight $w(x/q)$ is still positive on $[-1, 1]$), the results on the limit behaviour (3.18) still hold, and the singularity confinement property is still valid as well. Even for $c \geq 1$, where for small n we observe that y_n can be negative, we still observe that y_n is bounded and oscillating, however the even and odd subsequences do not tend to q^α and 1 respectively anymore, but to q^α/c and $1/c$, respectively.*

5 Connection to α-q-P_{V}

It is worth noticing that the obtained recurrence relations can be seen as a limiting case of the set of asymmetric discrete Painlevé equations α-q-P_{V}. This set can be found as the discrete Painlevé equation connected to the affine Weyl group E_6^q in Sakai's classification(see, e.g., [4]), or as α-q-P_{V} in [11]. The equations are

$$(1 - u_n v_n)(1 - u_n v_{n-1}) = \frac{(u_n - 1/p)(u_n - 1/r)(u_n - 1/s)(u_n - 1/t)}{(u_n - b\rho_n)(u_n - \rho_n/b)}$$

$$(1 - u_n v_n)(1 - u_{n+1} v_n) = \frac{(v_n - p)(v_n - r)(v_n - s)(v_n - t)}{(v_n - a w_n)(v_n - w_n/a)}$$

with $prst = 1$. Consider this system with the particular choice of parameters

$$p = 1, r = c, s = \kappa, t = \frac{1}{c\kappa}, b = c\kappa, \rho_n = q^{2n}, a = \kappa, w_n = q^{2n+\alpha+1}.$$

Then, letting κ tend to 0, we obtain as a limit the set of equations (4.5).

6 A stable method for computing the recurrence coefficients

A more stable way for computing the recurrence values is by writing (4.4) as a system of quadratic equations in $\tilde{u}_n := y_{2n}$ and $\tilde{v}_n := y_{2n+1}$, in the same way as we obtained (4.5):

$$\begin{cases} \tilde{u}_n^2 q^{-\alpha}\left(c^2 q^{2n}\tilde{v}_n\tilde{v}_{n-1} - c\right) + \tilde{u}_n\left(-cq^{2n}(\tilde{v}_n + \tilde{v}_{n-1}) + c + 1\right) + q^{\alpha}\left(q^{2n} - 1\right) &= 0 \\ \tilde{v}_n^2\left(c^2 q^{2n+1-\alpha}\tilde{u}_n\tilde{u}_{n+1} - c\right) + \tilde{v}_n\left(-cq^{2n+1}(\tilde{u}_n + \tilde{u}_{n+1}) + c + 1\right) + \left(q^{2n+1+\alpha} - 1\right) &= 0. \end{cases}$$

Computing the discriminants and opting for their positive roots (as we know, the $\tilde{u}_n$ and $\tilde{v}_n$ are related to the recurrence coefficients which are positive) in the expression of $\tilde{u}_n$ and $\tilde{v}_n$, we find

$$\begin{cases} \tilde{u}_n &= f_n(-cq^{2n}(\tilde{v}_n + \tilde{v}_{n-1}) + c + 1, c^2 q^{2n}\tilde{v}_n\tilde{v}_{n-1} - c) \\ \tilde{v}_n &= g_n(-cq^{2n+1}(\tilde{u}_n + \tilde{u}_{n+1}) + c + 1, c^2 q^{2n+1-\alpha}\tilde{u}_n\tilde{u}_{n+1} - c) \end{cases}$$

with

$$\begin{cases} f_n(x,y) &= \frac{-x+\sqrt{x^2+4(1-q^{2n})y}}{2q^{-\alpha}y} \\ g_n(x,y) &= \frac{-x+\sqrt{x^2+4(1-q^{2n+1+\alpha})y}}{2y}. \end{cases}$$

We now define an operator T, acting on the space of double non-negative rows:

$$T\begin{pmatrix} \xi &= (\xi_0, \xi_1, \ldots) \\ \eta &= (\eta_0, \eta_1, \ldots) \end{pmatrix} = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$$

where

$$\xi'_0 = 0,\ \ \xi'_n = f_n(-cq^{2n}(\eta_n + \eta_{n-1}) + c + 1, c^2 q^{2n}\eta_n\eta_{n-1} - c) \text{ for } n > 0$$

and

$$\eta'_n = g_n(-cq^{2n+1}(\xi_n + \xi_{n+1}) + c + 1, c^2 q^{2n+1-\alpha}\xi_n\xi_{n+1} - c) \text{ for } n \geq 0.$$

The solution of the equation which comes from our recurrence coefficients, denoted by $(\tilde{u}, \tilde{v})$, now coincides with a fixed point of this operator T. We know that $(\tilde{u}, \tilde{v}) \geq (0,0)$ (inequalities should be interpreted termwise: as holding between any two elements on corresponding positions). Unleashing the operator T (on any inequality we will use) reverses inequalities.

Lemma 6.1. *If $(\xi, \eta) \leq (a, b)$, then $T(\xi, \eta) \geq T(a, b)$.*

Proof. It is not hard to check that all partial derivatives of f_n and g_n are negative in the region where we need them. Which region this is, depends on the parameter c. $\square$

We now construct the following sequence by repeatedly applying T to $(0,0)$. This sequence has increasing subsequence $T^{2k}(0,0)$ and decreasing subsequence $T^{2k+1}(0,0)$. We denote their limits by (ξ^-, η^-) and (ξ^+, η^+), respectively. By continuity of T we get that $T(\xi^-, \eta^-) = (\xi^+, \eta^+)$ and $T(\xi^+, \eta^+) = (\xi^-, \eta^-)$. Furthermore, any fixed point (ξ^*, η^*) with positive components has to obey $(\xi^-, \eta^-) \leq (\xi^*, \eta^*) \leq (\xi^+, \eta^+)$. If one is able to show that $(\xi^-, \eta^-) = (\xi^+, \eta^+)$, one proves there is only one fixed point, which necessarily coincides with the solution of the discrete Painlevé equation coming from the recurrence coefficients. We did not find this proof (yet), but numerical experiments strongly suggest that indeed $(\xi^-, \eta^-) = (\xi^+, \eta^+)$. This would also be enough to prove the uniqueness of positive solutions to (4.4) or (3.16) with $y_0 = 0$, as mentioned in Section 3. See Figure 3 for the result using a particular choice of parameters, and Figure 4 for a comparison between the two methods.

Remark 6.2. *If $c = 0$ then (4.4) immediately gives the exact recurrence coefficients: $y_n = q^{\alpha} - q^{n+2\alpha}$ for even n, and $y_n = 1 - q^{n+\alpha}$ for odd n, which gives*

$$a_n^2 = \begin{cases} q^{n+\alpha-1}(1 - q^{n+\alpha}) & \text{for even } n \\ q^{n-1}(1 - q^{n+\alpha}) & \text{for odd } n. \end{cases}$$

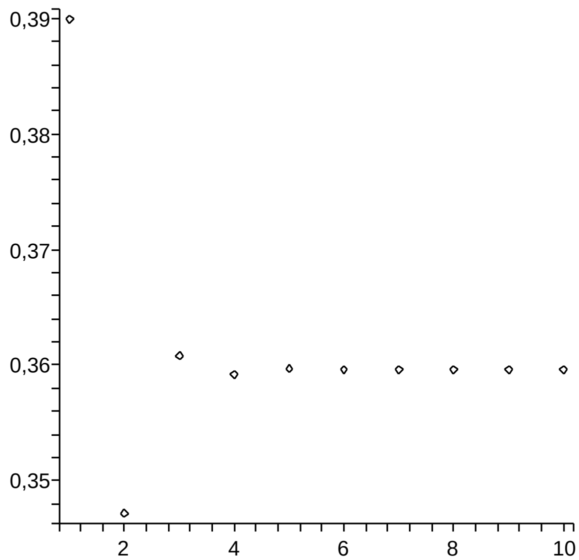

Figure 3: Approximation of the recurrence coefficient y_1 for $\alpha = 2, q = 9/10, c = -1/2$; on the horizontal axis is the number of times we applied the operator T to the starting value $(0, 0)$.

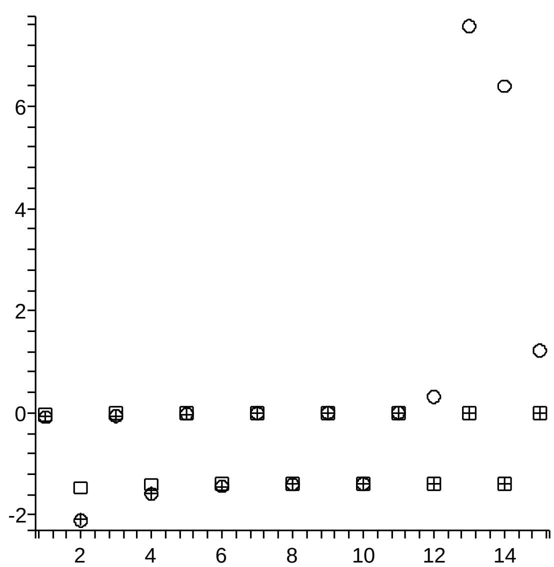

Figure 4: $\log |y_n|$ with $\alpha = 2, q = 1/2, c = -5/2$, computed using the recurrence relations (4.4) with 20 digits accuracy (circles), and the approximation obtained after applying once (squares), or three times (crosses) the operator T of Section 6 to the starting value $(0, 0)$.

7 Conclusions

We used a generalized q-Freud weight in order to find a q-discrete Painlevé equation. What we found was an asymmetric form of d-$\mathrm{P_I}$ which has (as we believe) never been described before. The asymmetric form emerges in a very natural way from the parity of the orthogonal polynomials associated to the weight. We showed its relation to $\alpha - q\mathrm{P_V}$ and gave a stable method to compute the recurrence coefficients for the orthonormal polynomials associated to this weight.

The technique used in this paper is not sufficient to handle weights that are not even. In these cases, one has a non-zero b_n entering the system, which could lead to new results. This is a possible interest for future research.

8 Acknowledgements

This research was supported by Belgian Interuniversity Attraction Poles Programme P6/02, K.U.Leuven projects OT/04/21 and OT/08/33.

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